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1998 J. Phys. A: Math. Gen. 31 5289

(http://iopscience.iop.org/0305-4470/31/23/012)

J. Phys. A: Math. Gen. **31** (1998) 5289–5298. Printed in the UK PII: S0305-4470(98)90656-4

Braided chains of q-deformed Heisenberg algebras

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Received 13 January 1998

Abstract. Given M copies of a q-deformed Weyl or Clifford algebra in the defining representation of a quantum group G_q, we determine a prescription to embed them into a unique, inclusive G_q-covariant algebra. The different copies are 'coupled' to each other and are naturally ordered into a 'chain'. In the case $G_q = SL_q(N)$ a modified prescription yields an inclusive algebra which is even explicitly $GL_q(M) \times SL_q(N)$-covariant, where $SL_q(M)$ is a symmetry relating the different copies. By the introduction of these inclusive algebras we significantly enlarge the class of G_q-covariant deformed Weyl/Clifford algebras available for physical applications.

1. Introduction

Weyl and Clifford algebras (respectively denoted by $\mathcal{A}_+$, $\mathcal{A}_-$ and collectively as 'Heisenberg algebras') are at the heart of quantum physics. The most useful Heisenberg algebras are those endowed with definite transformation properties under the action of some symmmetry Lie group G (or Lie algebra $\boldsymbol{g}$).

The idea that quantum groups [1] could generalize Lie groups in describing symmetries of quantum physical systems has attracted much interest in the past decade. Mathematically speaking, a quantum group can be described as a deformation $\mathrm{Fun}(G_q)$ of the algebra $\mathrm{Fun}(G)$ of regular functions on G or, in the dual picture, as a deformation $U_q\boldsymbol{g}$ of the universal enveloping algebra $U\boldsymbol{g}$, within the category of (quasitriangular) Hopf algebras; here $q = \mathrm{e}^h$, and h is the deformation parameter. These q-deformations induce matched q-deformations of all $\mathrm{Fun}(G_q)$-comodule algebras (i.e. of the algebras whose generators satisfy commutation relations that are preserved by the $\mathrm{Fun}(G_q)$-coaction), in particular of G-covariant Heisenberg algebras. q-deformed Heisenberg algebras corresponding to a simple Lie algebra $\boldsymbol{g}$ in the classical series A_n, B_n, D_n were introduced in [2–5] in the restricted case that the generators A_i^+, A^i belong respectively to the defining corepresentation ϕ_d of $\mathrm{Fun}(G_q)$ and to its contragradient $\phi_d^\vee$.

In general, we shall denote by $\mathcal{A}_{\pm,G,\phi}$ the Weyl/Clifford algebra with generators a^i, a_i^+ belonging respectively to some corepresentation ϕ of G and to its contragradient $\phi^\vee$ and fulfilling the canonical (anti)commutation relations

$$a_i^+ a_j^+ \mp a_j^+ a_i^+ = 0 \tag{1.1}$$

$$a^i a^j \mp a^j a^i = 0 \tag{1.2}$$

$$a^i a_j^+ - \delta_j^i \mathbf{1} \mp a_j^+ a^i = 0. \tag{1.3}$$

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The purpose of this work is to find out whether there exists some G_q-covariant deformation of $\mathcal{A}_{\pm,G,\phi}$ (which we will denote by $\mathcal{A}^q_{\pm,G,\phi}$) having the same Poincaré series as $\mathcal{A}_{\pm,G,\phi}$. We shall denote the generators of $\mathcal{A}^q_{\pm,G,\phi}$ by A^i, A^+_i.

As a preliminary result we show (section 3) that, besides $\mathcal{A}^q_{\pm,SL(N),\phi_d}$†, $\mathcal{A}^q_{+,SO(N),\phi_d}$ [2–5], $\mathcal{A}^q_{-,Sp(n),\phi_d}$ can also be defined. The first major result is, however, that one can embed M identical copies of $\mathcal{A}^q_{+,G,\phi_d}$ (resp. $\mathcal{A}^q_{-,G,\phi_d}$) into a unique, well-defined algebra $\mathcal{A}^q_{+,G,\phi_M}$ (resp. $\mathcal{A}^q_{-,G,\phi_M}$), or more generally $M' < M$ copies of $\mathcal{A}^q_{+,G,\phi_d}$ and $(M - M')$ copies of $\mathcal{A}^q_{-,G,\phi_d}$ into a unique, well-defined deformed superalgebra $\mathcal{A}^q_{G,\phi_M}$; ϕ_M denotes here the direct sum of M copies of ϕ_d. Due to the rules of braiding [6], the different copies do not commute with each other; consistent commutation relations between the latter require the introduction of an ordering: we call the ordered sequence a 'braided chain'.

The use of the symbols a^i, a^+_i, A^i, A^+_i etc does not necessarily mean that we are dealing with creators and annihilators; the latter fact is rather determined by the choice of the $*$-structure, if any. In section 4 we consider the natural $*$-structures giving the generators the meaning of creation and annihilation operators, or for example of coordinates and derivatives.

The second major result (section 5) is that if $G_q = SL_q(N)$ one can modify the A–A^+ commutation relations of $\mathcal{A}^q_{\pm,SL(N),\phi_M}$ in such a way that the generators become explicitly $GL_q(M) \times SL_q(N)$-covariant‡. The additional symmetry $GL_q(M)$ transforms the different copies into each other, as in the classical case.

The physical relevance of the case that ϕ is a direct sum of many copies of ϕ_d's is easily understood once one notes that the different copies could correspond to different particles, crystal sites or space(time)-points, respectively in quantum mechanics, condensed matter physics or quantum field theory. The coupling (i.e. noncommutativity) between the different copies can be interpreted as a naturally built-in form of interaction between them. In the particular case that $\mathcal{A}^q_{\pm,G,\phi}$ (with $q \in \mathbb{R}$) is a q-deformation of the $*$-algebra $\mathcal{A}_{\pm,G,\phi}$ with $(a^i)^\dagger = a^+_i$, then the physical interpretation of A^i, A^+_j as annihilators and creators does not necessarily requires the introduction of particles with *exotic* statistics. Indeed, it is possible to adopt ordinary boson/fermion statistics [8, 9], whereby A^+_i, A^i are to be interpreted as 'composite operators' creating and destroying some sort of 'dressed states' of bosons/fermions.

2. Preliminaries

For a simple Lie group G the algebra $\mathrm{Fun}(G_q)$ [10] is generated by N^2 objects T^i_j, $i, j = 1, \ldots, N$, fulfilling the commutation relations

$$\hat{R}^{ij}_{hk} T^h_l T^k_m = T^i_h T^j_k \hat{R}^{hk}_{lm}. \tag{2.1}$$

N is the dimension of the defining representation of G, $\hat{R}$ the corresponding 'braid matrix' [10], i.e. a numerical matrix fulfilling the 'braid equation'

$$\hat{R}_{12}\hat{R}_{23}\hat{R}_{12} = \hat{R}_{23}\hat{R}_{12}\hat{R}_{23}. \tag{2.2}$$

Here we have used the conventional tensor notation $(M_{12})^{ijk}_{lmn} = M^{ij}_{lm}\delta^k_n$, etc. Because of equations (2.2) and (2.1) $\mathrm{Fun}(G_q)$ is also a bialgebra with coproduct and counit respectively given by $\Delta(T^i_j) = T^i_h \otimes T^h_j$ and $\varepsilon(T^i_j) = \delta^i_j$.

† $SL(N)$ can be easily promoted also to a $GL(N)$.

‡ The result regarding $\mathcal{A}^q_+$, $SL(N)$, ϕ_M was essentially already found in [7], whose author we thank for drawing our attention to this point.

A (right) comodule algebra of $\mathrm{Fun}(G_q)$ is an algebra $\mathcal{M}$ equipped with a 'corepresentation' ϕ, i.e. with an algebra homomorphism $\phi : \mathcal{M} \to \mathcal{M} \otimes \mathrm{Fun}(G_q)$ such that $(\mathrm{id} \otimes \Delta) \circ \phi = (\phi \otimes \mathrm{id}) \circ \phi$. For any polynomial function $f(t)$ in one variable, the algebra $\mathcal{M}$ generated by N objects A_i^+ fulfilling the quadratic relations

$$[f(\hat{R})]^{ij}_{hk} A_i^+ A_j^+ = 0 \tag{2.3}$$

and equipped with the algebra homomorphism $\phi_d(A_i^+) := A_j^+ \otimes T_i^j$ is a comodule algebra [10].

By adding to the quadratic conditions (2.1) some suitable inhomogeneous condition [10], $\mathrm{Fun}(G_q)$ can be endowed also with an antipode S and therefore becomes a Hopf algebra†. Then the algebra $\mathcal{M}'$ generated by N objects A^i fulfilling the quadratic relations

$$[f(\hat{R})]^{hk}_{ij} A^j A^i = 0 \tag{2.4}$$

and equipped with the algebra homomorphism $\phi_d^\vee(A^i) := A^j \otimes ST^i_j$ is a comodule algebra with inverse transformation properties of $\mathcal{M}$; therefore the corepresentation $\phi_d^\vee$ can be called the contragradient of ϕ_d.

To continue, we need to recall some specific information regarding each quantum group G_q. The braid matrix $\hat{R}$ of the quantum group $\mathrm{Fun}(G_q)$ is a $N^2 \times N^2$ matrix that admits the following projector decomposition [10]

$$\begin{aligned}
&\hat{R} = q\mathcal{P}^S - q^{-1}\mathcal{P}^A \qquad \text{if } G = SL(N)\\
&\hat{R} = q\mathcal{P}^s - q^{-1}\mathcal{P}^a + q^{1-N}\mathcal{P}^t \qquad \text{if } G = SO(N)\\
&\hat{R} = q\mathcal{P}^{s'} - q^{-1}\mathcal{P}^{a'} - q^{1-N}\mathcal{P}^{t'} \qquad \text{if } G = Sp(n),\ N = 2n
\end{aligned} \tag{2.5}$$

with

$$\mathcal{P}^\mu \mathcal{P}^\nu = \delta^{\mu\nu} \qquad \sum_\mu \mathcal{P}^\mu = 1. \tag{2.6}$$

$\mathcal{P}^A, \mathcal{P}^S$ are $SL_q(N)$-covariant q-deformations of the antisymmetric and symmetric projectors respectively; they have dimensions $\frac{N(N-1)}{2}$ and $\frac{N(N+1)}{2}$ respectively. $\mathcal{P}^a, \mathcal{P}^t, \mathcal{P}^s$ are $SO_q(N)$-covariant q-deformations of the antisymmetric, trace, and symmetric trace-free projectors respectively; they have dimensions $\frac{N(N-1)}{2}$, 1 and $\frac{N(N+1)}{2} - 1$ respectively. $\mathcal{P}^{s'}, \mathcal{P}^{t'}, \mathcal{P}^{a'}$ are $Sp_q(n)$-covariant ($N = 2n$) q-deformations respectively of the symmetric, symplectic, antisymmetric symplectic-free projectors; they have dimensions $\frac{N(N+1)}{2}$,1 and $\frac{N(N-1)}{2} - 1$ respectively. Setting

$$\begin{aligned}
&\mathcal{P}^+ = \mathcal{P}^S \qquad \text{if } G = SL(N)\\
&\mathcal{P}^+ = \mathcal{P}^s + \mathcal{P}^t \qquad \text{if } G = SO(N)\\
&\mathcal{P}^+ = \mathcal{P}^{s'} \qquad \text{if } G = Sp(n)\\
&\mathcal{P}^- = \mathcal{P}^A \qquad \text{if } G = SL(N)\\
&\mathcal{P}^- = \mathcal{P}^a \qquad \text{if } G = SO(N)\\
&\mathcal{P}^- = \mathcal{P}^{a'} + \mathcal{P}^{t'} \qquad \text{if } G = Sp(n)
\end{aligned} \tag{2.7}$$

we obtain $\mathrm{Fun}(G_q)$-covariant deformations $\mathcal{P}^+, \mathcal{P}^-$ of the $\frac{N}{2}(N+1)$-dim symmetric and $\frac{N}{2}(N-1)$-dim antisymmetric projector respectively.

† In the case $G_q = SL_q(N)$ this condition reads $\det_q T = 1$, where $\det_q T$ is the q-deformed determinant of T. One can also define a Hopf algebra $GL_q(N)$ by using the same $\hat{R}$-matrix, introducing a new generator t that is central and group-like, together with its inverse t^{-1}, and then imposing the weaker condition $\det_q T = t$.

In the following we shall also need the explicit expression for the $\hat{R}$ matrix of $SL_q(N)$ and for its inverse:

$$\hat{R} = q\sum_{i=1}^{N} e_i^i \otimes e_i^i + \sum_{i\neq j} e_i^j \otimes e_j^i + (q-q^{-1})\sum_{i<j} e_i^i \otimes e_j^j. \tag{2.8}$$

$$\hat{R}^{-1} = q^{-1}\sum_{i=1}^{N} e_i^i \otimes e_i^i + \sum_{i\neq j} e_i^j \otimes e_j^i + (q^{-1}-q)\sum_{i>j} e_i^i \otimes e_j^j. \tag{2.9}$$

Here we have used the conventional tensor notation and denoted by e_j^i the $N \times N$ matrix with $(e_j^i)_k^h = \delta^{ih}\delta_{jk}$.

By repeated application of the equations (2.2), (2.1) we find

$$\begin{aligned} & f(\hat{R})^{ij}_{hk} T^h_l T^k_m = T^i_h T^j_k f(\hat{R})^{hk}_{lm} \\ & f(\hat{R}_{12})\hat{R}_{23}\hat{R}_{12} = \hat{R}_{23}\hat{R}_{12} f(\hat{R}_{23}) \end{aligned} \tag{2.10}$$

for any polynomial function $f(t)$ in one variable, in particular for those f's yielding $f(\hat{R}) = \mathcal{P}^\mu$ or $f(\hat{R}) = \hat{R}^{-1}$. Equations (2.2), (2.1) and (2.10) are evidently satisfied also after the replacement of $\hat{R} \to \hat{R}^{-1}$.

If in relations (2.3) and (2.4) one chooses $f(\hat{R}) = \mathcal{P}^\mp$, these equations become the $\mathrm{Fun}(G_q)$-covariant deformations of the (anti)commutation relations (1.1) and (1.2):

$$\mathcal{P}^{\mp}{}^{ij}_{hk} A^+_i A^+_j = 0 \tag{2.11}$$

$$\mathcal{P}^{\mp}{}^{ij}_{hk} A^k A^h = 0. \tag{2.12}$$

Relations (1.1), (1.2), (2.11) and (2.12) amount each to $\frac{N(N-1)}{2}$ or to $\frac{N(N+1)}{2}$ independent relations, respectively if the upper or the lower sign is considered. The algebras $\mathcal{M}$, $\mathcal{M}'$ defined resp. by (2.11) and (2.12) have [10, 11] the same Poincaré series as the algebras defined by resp. by (1.1) and (1.2).

To obtain $\mathrm{Fun}(G_q)$-covariant deformations $\mathcal{A}^q_{\pm,G,\phi_d}$ of the classical Heisenberg algebras described in section 1 one still has to deform relations (1.3). For $G_q = SL_q(N), SO_q(N)$ this was done in [2–5]. The natural ansatz is to look for quadratic cross commutation relations, in the form

$$A^i A^+_j = \delta^i_j \mathbf{1} \pm S^{ih}_{jk} A^+_h A^k. \tag{2.13}$$

The inhomogeneous term is fixed by the requirement that $\{A^i\}$ is the basis dual to $\{A^+_i\}$. The numerical matrix S has to be determined imposing $\mathrm{Fun}(G_q)$-covariance and that $\mathcal{A}^q_{\pm,G,\phi_d}$ itself has the same Poincaré series as its classical counterpart $\mathcal{A}_{\pm,G,\phi_d}$. It will be convenient to use the following general lemma.

Lemma 1. Let $\hat{R} = \sum_\mu c_\mu \mathcal{P}^\mu$ be the projector decomposition of the braid matrix $\hat{R}$, and let $\mathcal{P}^+ := \sum_{\mu:\, c_\mu>0} \mathcal{P}^\mu$ and $\mathcal{P}^- := \sum_{\mu:\, c_\mu<0} \mathcal{P}^\mu$ be the corresponding deformed symmetric and antisymmetric projectors respectively. Assume that relations (2.11) and (2.12) define algebras $\mathcal{M}$, $\mathcal{M}'$ with the same Poincaré series as their classical counterparts. In order that relations (2.11)–(2.13) define a deformed Weyl algebra $\mathcal{A}^q_+$ (resp. Clifford algebra $\mathcal{A}^q_-$) with the same Poincaré series as its classical counterpart $\mathcal{A}_+$ (resp. $\mathcal{A}_-$) there must exist exactly one negative (resp. positive) c_μ, say c_- (resp. c_+), and the commutation relations (2.13) have to take one of the two following forms

$$A^i A^+_j = \delta^i_j \mathbf{1} \pm (c_\mp)^{-1} \hat{R}^{ih}_{jk} A^+_h A^k \tag{2.14}$$

$$A^i A^+_j = \delta^i_j \mathbf{1} \pm c_\mp \hat{R}^{-1}{}^{ih}_{jk} A^+_h A^k. \tag{2.15}$$

Proof. Let us multiply equation (2.11) by A^l from the left. We easily find

$$0 = A^l \mathcal{P}^{\mp}{}^{ij}_{hk} A^+_i A^+_j$$
$$\stackrel{(2.13)}{=} [\mathcal{P}^{\pm}(\mathbf{1}+S)]^{li}_{hk} A^+_i + (S_{12}S_{23}\mathcal{P}^{\pm}_{12})^{lij}_{hkm} A^+_i A^+_j A^m.$$

To ensure that the second term vanishes without introducing new, third degree relations (which would yield a different Poincaré series) it must be either $S \propto \hat{R}$ or $S \propto \hat{R}^{-1}$, so that

$$(S_{12}S_{23}\mathcal{P}^{\pm}_{12})^{lij}_{hkm} A^+_i A^+_j A^m \stackrel{(2.10)}{=} (\mathcal{P}^{\pm}_{23}S_{12}S_{23})^{lij}_{hkm} A^+_i A^+_j A^m \stackrel{(2.11)}{=} 0.$$

These correspond to the two possible braidings [6]. If $S = b\hat{R}$, then the first term vanishes iff

$$0 = \mathcal{P}^{\pm}(\mathbf{1}+S) = \sum_{\mu:\pm c_\mu > 0} \mathcal{P}^{\mu}(\mathbf{1}+c_\mu b) \Leftrightarrow \mathbf{1}+c_\mu b \qquad \forall \mu : \pm c_\mu > 0.$$

Thus there must exist exactly one positive (resp. negative) c_μ and relation (2.14) must hold. Similarly one proves relation (2.15) if $S = b\hat{R}^{-1}$. □

As immediate consequences of this lemma and of the decompositions (2.7) we find the following.

- There exists no satisfactory definitions of the q-deformed algebras $\mathcal{A}^q_{-,SO(N),\phi_d}$, $\mathcal{A}^q_{+,sp(n),\phi_d}$, since these correspond respectively to the projectors $(2.7)_2$, $(2.7)_6$.
- There exists satisfactory definitions of the q-deformed algebras $\mathcal{A}^q_{+,SL(N),\phi_d}$ [2, 3], $\mathcal{A}^q_{-,SL(N),\phi_d}$ [4], $\mathcal{A}^q_{+,SO(N),\phi_d}$ [5], $\mathcal{A}^q_{-,sp(n),\phi_d}$, since these are the algebras corresponding respectively to the projector $(2.7)_4$ $(2.7)_1$, $(2.7)_5$, $(2.7)_3$ (to our knowledge, the latter has never been considered before in the literature).

3. Main embedding prescription

We would like to generalize the construction of the preceding section to the case in which A^+_i, A^i belong respectively to corepresentations $\phi_M, \phi^\vee_M$ that are direct sums of $M \geqslant 1$ copies of $\phi_d, \phi^\vee_d$. Let ${}^{\alpha}\mathcal{A}^q_{\pm,G,\phi_d}$ ($\alpha = 1, 2, \ldots, M$) be G_q-covariant q-deformed Heisenberg algebra with generators $\mathbf{1}$, $A^{\alpha,i}$, $A^+_{\alpha,i}$, $i = 1, \ldots, N$, and relations

$$\mathcal{P}^{(\alpha)}{}^{hk}_{ij} A^+_{\alpha,h} A^+_{\alpha,k} = 0 \tag{3.1}$$
$$\mathcal{P}^{(\alpha)}{}^{ij}_{hk} A^{\alpha,k} A^{\alpha,h} = 0 \tag{3.2}$$
$$A^{\alpha,i} A^+_{\alpha,j} - \delta^i_j \mathbf{1} - (-1)^{\epsilon_\alpha}[(q^{1-2\epsilon_\alpha}\hat{R})^{\eta_\alpha}]^{ih}_{jk} A^+_{\alpha,h} A^{\alpha,k} = 0. \tag{3.3}$$

According to the last remark in the previous section, let ϵ_α take the values $\epsilon_\alpha \equiv 0$ if $G = SO(N)$, $\epsilon_\alpha \equiv 1$ if $G = Sp(n)$, and $\epsilon_\alpha = 0, 1$ if $G = SL(N)$; $\epsilon_\alpha = 0, 1$ correspond to Weyl and Clifford respectively. Moreover, let

$$\mathcal{P}^{(\alpha)} = \begin{cases} \mathcal{P}^+ & \text{if } \epsilon_\alpha = 0 \\ \mathcal{P}^- & \text{if } \epsilon_\alpha = 1. \end{cases} \tag{3.4}$$

Recalling that the comodules of $\mathrm{Fun}(G_q)$ belong to a braided monoidal category, we know that consistent commutation relations between the generators of ${}^{\alpha}\mathcal{A}^q_{\pm,G,\phi_d}$, ${}^{\beta}\mathcal{A}^q_{\pm,G,\phi_d}$, $\alpha \neq \beta$, are given by the two possible braidings (the latter correspond to the quasitriangular structures $\mathcal{R}, \mathcal{R}_{21}^{-1}$ [6]). Accordingly, the commutation relations between $A^+_{\alpha,i}, A^+_{\beta,j}$ for instance may become

$$\text{either} \qquad A^+_{\alpha,i}, A^+_{\beta,j} \propto \hat{R}^{hk}_{ij} A^+_{\beta,h} A^+_{\alpha,k}$$
$$\text{or} \qquad A^+_{\alpha,i}, A^+_{\beta,j} \propto \hat{R}^{-1}{}^{hk}_{ij} A^+_{\beta,h} A^+_{\alpha,k}.$$

There are $\frac{M(M-1)}{2}$ different pairs (α, β); if we could choose for each pair the upper or lower solution independently we would have $2^{\frac{M(M-1)}{2}}$ different versions of the deformed commutation relations. We claim that, in fact, only $M!$ are allowed, in other words that, up to a reordering (i.e. a permutation of the α's), the only consistent way is as follows.

Proposition 1. Without loss of generality we can assume that

$$A^+_{\alpha,i}, A^+_{\beta,j} = (-1)^{\epsilon_\alpha \epsilon_\beta} c_{\alpha\beta} \hat{R}^{hk}_{ij} A^+_{\beta,h} A^+_{\alpha,k} \qquad \text{if } \alpha < \beta \tag{3.5}$$

with $c_{\alpha\beta} \overset{q\to 1}{\to} 1$.

(We have factorized the overall sign necessary to obtain the correct commutation relations between fermionic or bosonic variables in the classical limit).

Proof. The claim can be proved inductively. It is obvious if $M = 2$. Assume now that the claim is true when $M = P$, and call $A^+_{\cdot,i}$ the generators of the $(P + 1)$th additional subalgebra. We need to prove that

$$A^+_{\beta,i} A^+_{\cdot,j} \propto \hat{R}^{hk}_{ij} A^+_{\cdot,h} A^+_{\beta,k} \Rightarrow A^+_{\alpha,i} A^+_{\cdot,j} \propto \hat{R}^{hk}_{ij} A^+_{\cdot,h} A^+_{\alpha,k} \qquad \forall \alpha < \beta$$
$$A^+_{\gamma,i} A^+_{\cdot,j} \propto \hat{R}^{-1}{}^{hk}_{ij} A^+_{\cdot,h} A^+_{\gamma,k} \Rightarrow A^+_{\delta,i} A^+_{\cdot,j} \propto \hat{R}^{-1}{}^{hk}_{ij} A^+_{\cdot,h} A^+_{\delta,k} \qquad \forall \delta > \gamma.$$

Let $A^+_{\beta,i} A^+_{\cdot,j} = V^{hk}_{ij} A^+_{\cdot,h} A^+_{\beta,k}$; we can invert the order of the factors in the product $A^+_{\alpha,h} A^+_{\beta,i} A^+_{\cdot,j}$ either by permuting the first two factors, then the last two, finally the first two again, or by permuting the last two factors, then the first two, finally the last two again; to get the same result we need that $\hat{R}_{12} V_{23} \hat{R}_{12} = \hat{R}_{23} V_{12} \hat{R}_{23}$. This equation is satisfied iff $V \propto \hat{R}$. Thus we have proved the first implication. Similarly one proves the second. $\square$

Equation (3.5) and the condition that $A^{\alpha,i}$ are the dual generators of $A^+_{\alpha,i}$ implies (for $\alpha < \beta$)

$$A^{\alpha,j} A^{\beta,i} = (-1)^{\epsilon_\alpha \epsilon_\beta} c_{\alpha\beta} \hat{R}^{ij}_{hk} A^{\beta,k} A^{\alpha,h}. \tag{3.6}$$

As for the remaining relations, we shall look for them in the form $A^{\beta,i} A^+_{\alpha,j} = M^{ih}_{jk} A^+_{\alpha,h} A^{\beta,k}$. It is easy to check that from either of the previous relation and the commutation relations of ${}^{\alpha}\mathcal{A}^q_{\pm,G,\phi_d}$ it follows (for $\alpha < \beta$):

$$A^{\beta,i} A^+_{\alpha,j} = (-1)^{\epsilon_\alpha \epsilon_\beta} c_{\alpha\beta} \hat{R}^{ih}_{jk} A^+_{\alpha,h} A^{\beta,k} \tag{3.7}$$

$$A^{\alpha,i} A^+_{\beta,j} = (-1)^{\epsilon_\alpha \epsilon_\beta} c^{-1}_{\alpha\beta} (\hat{R}^{-1})^{ih}_{jk} A^+_{\beta,h} A^{\alpha,k}. \tag{3.8}$$

For instance, relation (3.7) is derived by consistency when requiring that one gets the same result from $A^{\alpha,i} A^+_{\alpha,j} A^+_{\beta,k}$ either by permuting the first two factors, then the last two, finally the first two again, or by permuting the last two factors, then the first two and finally the last two again.

We will call $\mathcal{A}^q_{G,\phi_M}$ the unital algebra generated by $\mathbf{1}, A^{\alpha,i}, A^+_{\alpha,i}$, $\alpha = 1, 2, \ldots, M$, $i = 1, \ldots, N$ and commutation relations (3.1)–(3.3), (3.5), (3.7) and (3.8). We have thus proved the following proposition.

Proposition 2. $\mathcal{A}^q_{G,\phi_M}$ has the same Poincaré series as its classical counterpart $\mathcal{A}_{G,\phi_M}$.

4. $*$-structures

Let $\mathrm{Fun}(G_q)$ be a Hopf $*$-algebra and assume that ${}^{\alpha}\mathcal{A}^q_{\pm,G,\phi_d}$ are $\mathrm{Fun}(G_q)$-comodule $*$-algebras:

$$\phi_d(b^{\star_\alpha}) = [\phi_d(b)]^{\star_\alpha \otimes *} \qquad b \in {}^{\alpha}\mathcal{A}^q_{\pm,\mathfrak{g},\phi_d} \tag{4.1}$$

(here '$\star_\alpha$' denotes the $*$ of ${}^{\alpha}\mathcal{A}^q_{\pm,G,\phi_d}$). Can we use the $\star_\alpha$'s to build a $*$-structure $\star$ of the whole $\mathcal{A}^q_{\pm,G,\phi}$?

In the case that $*$ realizes the compact real section of $\mathrm{Fun}(G_q)$ (what requires $q \in \mathbb{R}^+$), then the simplest $*$-structure in $\mathcal{A}^q_{\pm,\mathfrak{g},\phi_d}$ is

$$(A^i)^\star = A^+_i. \tag{4.2}$$

It is immediate to check that the ansatz $(A^{i,\alpha})^\star = A^+_{i,\alpha}$ would be compatible with relations (3.1)–(3.3), but inconsistent with relations (3.5)–(3.8). Therefore let us choose the ansatz

$$(A^{i,\alpha})^\star = A^+_{i,\pi(\alpha)} \tag{4.3}$$

where π is some permutation of $(1, \ldots, M)$. It is easy to check that consistency with relations (3.5)–(3.8) requires

$$\pi(\alpha) = M - \alpha + 1 \tag{4.4}$$

$$\eta_{\pi(\alpha)} = \eta_\alpha \qquad c_{\pi(\alpha)\pi(\beta)} = c_{\beta\alpha} \qquad \epsilon_{\pi(\alpha)} = \epsilon_\alpha. \tag{4.5}$$

Equation (4.4) shows that π must be the inverse-ordering permutation; equation $(4.5)_3$ amounts to say that $\star$ must preserve the bosonic or fermionic character of the generators.

$\mathcal{A}^q_{+,SO(N),\phi_d}$ admits also an alternative $*$-structure compatible with ϕ_d, namely

$$(A^+_i)^\star = A^+_j C_{ji} \tag{4.6}$$

together with a nonlinear transformation for $(A^i)^\star$ [12]. Here C_{ij} is the q-deformed metric matrix [10], which is related to the projector $\mathcal{P}^t$ appearing in $(2.5)_2$ through the formula ${\mathcal{P}^t}^{ij}_{hk} = \frac{C^{ij}C_{hk}}{C^{lm}C_{lm}}$. It is easy to check that the ansatz

$$(A^+_{i,\alpha})^\star = A^+_{j,\pi(\alpha)} C^{ji} \tag{4.7}$$

together with the corresponding nonlinear one for $(A^{i,\alpha})^\star$, defines a consistent $*$-structure of $\mathcal{A}^q_{+,SO(N),\phi_d}$ provided relations (4.4) and (4.5) hold (with $\epsilon_\alpha \equiv 0 \ \forall\alpha$).

5. Modified prescription: $GL_q(M) \times G_q$-covariant algebras

If all the generators of $\mathcal{A}_{\pm,G,\phi}$ have the same Grassman parity, they belong to a corepresentation of $GL(M) \times G$. The coaction of the group $GL(M)$ amounts to a linear invertible transformation T of the $a^{\alpha,i}$ and of the $a^+_{\alpha,i}$:

$$a^{\alpha,i} \to a^{\beta,i} T^\alpha_\beta \qquad a^+_{\alpha,i} \to a^+_{\beta,i} {T^{-1}}^\beta_\alpha \tag{5.1}$$

which leaves the commutation relations (1.1)–(1.3) invariant. (If in addition we require some $*$-structure to be preserved, then T has to belong to some suitable subgroup of $GL(M)$ for example $T \in U(M)$ if $(a^i)^\dagger = a^+_i$.) We now try to construct a variant of the algebra of section 3 having explicitly $GL_q(M) \times G_q$-covariant generators†.

† Or equivalently $SL_q(M) \times G_q$-covariance, if we also impose the unit condition on the q-determinant of $GL_q(M)$.

Let T^α_β, $t = \det_q \|T^\alpha_\beta\|$ be the generators of the quasitriangular Hopf algebra Fun$[GL_q(M)]$, and T^a_b the generators of Fun(G_q) [10]. Let us introduce collective indices $A, B, \ldots$, denoting the pairs (α, a), $(\beta, b), \ldots$. The Hopf algebra Fun$(GL_q(M) \times G_q)$ can be defined as the algebra generated by objects T^A_B satisfying commutation relations which can be obtained from (2.1) by the replacement

$$T^A_B \to T^\alpha_\beta T^a_b \tag{5.2}$$

by assuming that $[T^\alpha_\beta, T^a_b] = 0$:

$$\hat{\mathbf{R}}^{AB}_{CD} T^C_E T^D_F = T^A_C T^B_D \hat{\mathbf{R}}^{CD}_{EF}. \tag{5.3}$$

Here $\hat{\mathbf{R}}$ is one of the matrices

$$\hat{\mathbf{R}}_{\pm}{}^{AB}_{CD} := \hat{R}_M^{\pm 1 \alpha\beta}{}_{\gamma\delta} \hat{R}^{ab}_{cd} \equiv (\hat{R}^{\pm 1}_M \otimes \hat{R})^{AB}_{CD} \tag{5.4}$$

and $\hat{R}_M$ is the braid matrix (2.8) of $SL_q(M)$. $\hat{\mathbf{R}}_\pm$ satisfies the braid equation, since $\hat{R}$, $\hat{R}_M$ do. The coproduct, counit, antipode and quasitriangular structure are introduced as in section 2 by $\Delta(T^A_B) = T^A_C \otimes T^C_B$, $\varepsilon(T^A_B) = \delta^A_B$, $ST^A_B = T^{-1}{}^A_B$.

A (right) comodule algebra of Fun$(GL_q(M) \times G_q)$ can be associated to the defining corepresentation of the latter, $\phi_D(A^+_A) = A^+_B \otimes T^B_A$, where A^+_C denote the generators. The dual comodule algebra, with generators A^C, will be associated to the contragradient corepresentation $\phi^\vee_D(A^A) = A^B \otimes ST^A_B$. To find compatible quadratic commutation relations among the A^+_B's (resp. A^B's) we need the projector decomposition of $\hat{\mathbf{R}}_\pm$, as in section 2. For this scope we just need to write down the projector decompositions of both $\hat{R}^{\pm 1}_M$ and $\hat{R}$ and note that $\mathbf{P} := \mathcal{P}_M \otimes \mathcal{P}'$ is a projector $\mathbf{P}$ whenever $\mathcal{P}, \mathcal{P}'$ are.

We start with the case $G_q = SL_q(N)$. We find

$$\begin{aligned}\hat{\mathbf{R}}_+ &= (q\mathcal{P}^S_M - q^{-1}\mathcal{P}^A_M) \otimes (q\mathcal{P}^S - q^{-1}\mathcal{P}^A)\\ &= -(\mathcal{P}^S_M \otimes \mathcal{P}^A + \mathcal{P}^A_M \otimes \mathcal{P}^S) + q^2\mathcal{P}^S \otimes \mathcal{P}^S + q^{-2}\mathcal{P}^A_M \otimes \mathcal{P}^A_M\\ &=: -\mathbf{P}^- + q^2\mathbf{P}^{S,1} + q^{-2}\mathbf{P}^{S,2}\end{aligned} \tag{5.5}$$

and

$$\begin{aligned}\hat{\mathbf{R}}_- &= (q^{-1}\mathcal{P}^S_M - q\mathcal{P}^A_M) \otimes (q\mathcal{P}^S - q^{-1}\mathcal{P}^A)\\ &= (\mathcal{P}^A_M \otimes \mathcal{P}^A + \mathcal{P}^S_M \otimes \mathcal{P}^S) - q^2\mathcal{P}^A_M \otimes \mathcal{P}^S - q^{-2}\mathcal{P}^S_M \otimes \mathcal{P}^A\\ &=: \mathbf{P}^+ - q^2\mathbf{P}^{A,1} - q^{-2}\mathbf{P}^{A,2}.\end{aligned} \tag{5.6}$$

We are in the condition to apply lemma 1. As a consequence, there exists a $GL_q(M) \times SL_q(N)$-covariant Weyl algebra $\mathcal{A}^q_{+,GL_q(M)\times SL_q(N),\phi_D}$, defined by the following commutation relations:

$$\mathbf{P}^-{}^{CD}_{AB} A^+_C A^+_D = 0 \tag{5.7}$$

$$\mathbf{P}^-{}^{AB}_{CD} A^D A^C = 0 \tag{5.8}$$

$$A^A A^+_B - \delta^A_B \mathbf{1} - \hat{\mathbf{R}}_+{}^{AC}_{BD} A^+_C A^D = 0. \tag{5.9}$$

Moreover, there exists a q-deformed $SL_q(M) \times SL_q(N)$-covariant Clifford algebra $\mathcal{A}^q_{-,SL_q(M)\times SL_q(N),\phi_D}$, defined by the following commutation relations:

$$\mathbf{P}^+{}^{CD}_{AB} A^+_C A^+_D = 0 \tag{5.10}$$

$$\mathbf{P}^+{}^{AB}_{CD} A^D A^C = 0 \tag{5.11}$$

$$A^A A^+_B - \delta^A_B \mathbf{1} + \hat{\mathbf{R}}_-{}^{AC}_{BD} A^+_C A^D = 0. \tag{5.12}$$

According to lemma 1, one could give also alternative definitions with $\hat{\mathbf{R}}^{-1}$ instead of $\hat{\mathbf{R}}$ in relations (5.9) and (5.12).

Let us verify that relations (5.7), (5.8), (5.10) and (5.11) are of the kind considered in section 3.

We take first relations (5.7) into account. We find

$$\begin{aligned}(q+q^{-1})^2\mathbf{P}^- &\overset{(5.5)}{=} (q+q^{-1})^2[\mathcal{P}^S\otimes\mathcal{P}^A+\mathcal{P}^A+\otimes\mathcal{P}^S]\\ &\overset{(2.7)_4}{=} (q\mathbf{1}-\hat{R}_M)\otimes(q^{-1}\mathbf{1}+\hat{R})+(q^{-1}\mathbf{1}+\hat{R}_M)\otimes(q\mathbf{1}-\hat{R})\\ &= 2(\mathbf{1}\otimes\mathbf{1}-\hat{R}_M\otimes\hat{R})+(q-q^{-1})(\mathbf{1}\otimes\hat{R}+\hat{R}_M\otimes\mathbf{1}).\end{aligned}$$

Using relation (2.8) we can write $\hat{R}_M$ explicitly and check that relations (5.7) amount to relations

$$\mathcal{P}^{-hk}_{\ ij}A^+_{\alpha,h}A^+_{\alpha,k}=0 \tag{5.13}$$

$$A^+_{\alpha,i},A^+_{\beta,j}-\hat{R}^{hk}_{ij}A^+_{\beta,h}A^+_{\alpha,k}=0 \qquad \text{if } \alpha<\beta. \tag{5.14}$$

Similarly one verifies that: (1) relations (5.8) amount to relations

$$\mathcal{P}^{-ij}_{\ hk}A^{\alpha,k}A^{\alpha,h}=0 \tag{5.15}$$

$$A^{\alpha,j}A^{\beta,i}-\hat{R}^{ij}_{hk}A^{\beta,k}A^{\alpha,h}=0 \qquad \text{if } \alpha<\beta; \tag{5.16}$$

(2) that relations (5.10) amount to relations†

$$\mathcal{P}^{+hk}_{\ ij}A^+_{\alpha,h}A^+_{\alpha,k}=0 \tag{5.17}$$

$$A^+_{\alpha,i},A^+_{\beta,j}+\hat{R}^{-1hk}_{\ \ \ ij}A^+_{\beta,h}A^+_{\alpha,k}=0 \qquad \text{if } \alpha<\beta; \tag{5.18}$$

(3) that relations (5.11) amount to relations

$$\mathcal{P}^{+ij}_{\ hk}A^{\alpha,k}A^{\alpha,h}=0 \tag{5.19}$$

$$A^{\alpha,j}A^{\beta,i}+\hat{R}^{-1ij}_{\ \ \ hk}A^{\beta,k}A^{\alpha,h}=0 \qquad \text{if } \alpha<\beta. \tag{5.20}$$

On the other hand, relations (5.9) and (5.12) for $\alpha\neq\beta$ are *not* of the type (3.7), (3.8) found in section 3; in fact, in a similar way one can show that relation (5.9) takes the form

$$A^{\alpha,a}A^+_{\beta,b}-\hat{R}^{ac}_{bd}A^+_{\beta,c}A^{\alpha,d}=0 \qquad \alpha\neq\beta \tag{5.21}$$

$$A^{\alpha,a}A^+_{\alpha,b}-\delta^a_b\mathbf{1}-q\hat{R}^{ac}_{bd}A^+_{\alpha,c}A^{\alpha,d}-(q-q^{-1})\sum_{\beta>\alpha}\hat{R}^{ac}_{bd}A^+_{\beta,c}A^{\beta,d}=0 \tag{5.22}$$

whereas relation (5.12) amounts to

$$A^{\alpha,a}A^+_{\beta,b}+\hat{R}^{ac}_{bd}A^+_{\beta,c}A^{\alpha,d}=0 \qquad \alpha\neq\beta \tag{5.23}$$

$$A^{\alpha,a}A^+_{\alpha,b}-\delta^a_b\mathbf{1}+q^{-1}\hat{R}^{ac}_{bd}A^+_{\alpha,c}A^{\alpha,d}-(q-q^{-1})\sum_{\beta<\alpha}\hat{R}^{ac}_{bd}A^+_{\beta,c}A^{\beta,d}=0. \tag{5.24}$$

Relations (5.21) and (5.23) specialized to the case $\alpha>\beta$ coincide with relations (3.7); specialized to the case $\alpha<\beta$, they differ from relations (3.8). Relations (5.22) and (5.24) differ from relations (3.3) by the additional terms with coefficient $(q-q^{-1})$.

The subalgebra $\mathcal{M}$ (resp. $\mathcal{M}'$) generated by A^+_A's (resp. A^A's) has the same Poincaré series of the subalgebra generated by classical $a^+_{\alpha a}$'s (resp. $a^{\alpha a}$'s), because of relations (5.13) and (5.14) (resp. (5.15) and (5.16)) in the Weyl case and because of relations (5.17) and (5.18) (resp. (5.19) and (5.20)) in the Clifford case. Since relations (5.9) and (5.12) allow to change the order of A^+_A's and A^B's in any product, we conclude with the following proposition.

† These are also of the type considered in section 3, provided we invert the order of Greek indices.

Proposition 3. The algebras $\mathcal{A}^{q}_{\pm,GL_q(M)\times SL_q(N),\phi_D}$ have the same Poincaré series as their classical counterparts.

Finally, let us ask about $*$-structures. When $q\in\mathbb{R}^+$ the Hopf algebra $GL_q(M)\times SL_q(N)$ admits the compact section $U_q(M)\times SU_q(N)$ [10]. The deformed Heisenberg algebras defined by relations (5.7)–(5.12) admit a natural $U_q(M)\times SU_q(N)$-covariant $*$-structure given by

$$(A^A)^\star = A^+_A \tag{5.25}$$

this can be easily checked by applying this $\star$ to relations (5.7)–(5.12) and by noting that $\hat{R}^T=\hat{R}$ and therefore $\hat{\mathbf{R}}^T=\hat{\mathbf{R}}$, $\mathbf{P}^T=\mathbf{P}$.

Let us now take into consideration the cases that $G_q=SO_q(N), Sp_q(n)$. The projector decomposition of $\hat{R}_M\otimes\hat{R}=\sum_\mu\lambda_\mu\mathbf{P}^\mu$ gives $\lambda_\mu=q^2,q^{-2},-1,\pm q^{2-N},\mp q^{-N}$, where the upper and lower sign refer to $G_q=SO_q(N)$ and $Sp_q(n)$ respectively. The projector decomposition of $\hat{R}_M^{-1}\otimes\hat{R}=\sum_\mu\lambda_\mu\mathbf{P}^\mu$ gives $\lambda_\mu=-q^2,-q^{-2},1,\mp q^{2-N}$, $\pm q^{-N}$. In both cases we always have more than one positive and more than one negative λ_μ. By lemma 1 no $GL_q(M)\times G_q$ covariant q-deformed Weyl/Clifford algebra can be built by this procedure.

Acknowledgments

It is a pleasure to thank J Wess for his scientific support and for the hospitality at his Institute. This work was financially supported through a TMR fellowship granted by the European Commission, Directorat General XII for Science, Research and Development, under the contract ERBFMICT960921.

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